CONVECTIVE MOTION IN THE SUPERCRITICAL RANGE FOR A SECOND-ORDER FLUID

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The evolution of supercritical perturbations in a horizontal layer of a viscoelastic fluid is analyzed, this layer being heated from below.

A study of convection in viscoelastic fluids has established that at definite values of the elasticity parameters there occurs an instability of the vibrational kind which does not occur under the same conditions in a Newtonian fluid [1]. Characteristic anomalies of viscoelastic fluids can also become evident during further evolution of perturbations into the supercritical range [2].

It has been demonstrated [3] that in a fluid describable by a Rivlin–Ericksen equation of state vibrational instability occurs before monotonic instability. It is interesting to examine the evolution of perturbations into the supercritical range for such a fluid. This study will deal with the effect which the elasticity of the fluid has on the evolution in time within the range of small supercriticality.

The rheological equations of state will be written in the form

$$\tau_{ij} = \mu a_{ij}^{(1)} + \beta a_{ik}^{(1)} a_{kj}^{(1)} + \gamma a_{ij}^{(2)}.$$

We now consider an infinitely large horizontal layer of thickness d heated from below.

The equations of convection in dimensionless form are

$$\partial_t u_i + P_{,i} + \mathrm{Ra}\theta\delta_{i3} - u_{i,jj} = S_{ij,j} - u_j u_{i,j}, \qquad (1)$$

$$\mathrm{Pr}\partial_t\theta - \theta_{,jj} - u_3 = -\mathrm{Pr}\, u_j\theta_{,j}. \qquad (2)$$

We consider the case of two free boundaries

$$\theta = u_3 = u_{3,33} = 0 \quad \text{at} \quad x_3 = \pm 1/2.$$

Here is a linear analysis of convective stability. Introduction of normal modes, according to the procedure in linear theory, yields the equation

$$(i\delta\mathrm{Pr} - \nabla^2)\,[i\delta - (1 + \gamma i\delta)\,\nabla^2]\,\nabla^2 W + \mathrm{Ra}\, a^2 W = 0 \qquad (3)$$

with the boundary conditions

$$W = W_{,33} = W_{,3333} = 0 \quad \text{at} \quad x_3 = \pm 1/2. \qquad (4)$$

Letting $W = \cos \pi x_3$, $E = \exp\,[i(kx_1 - lx_2 + \delta t)]$, we write the solution to Eqs. (3) and (4) as

$$u_3 = AWE,\ u_1 = \frac{ik}{a^2} AW_{,3}E,$$
$$u_2 = \frac{il}{a^2} AW_{,3}E,\ \theta = \frac{AWE}{i\delta\mathrm{Pr} + a^2 + \pi^2}, \qquad (5)$$

with $k^2 + l^2 = a^2$.

For a nonlinear analysis we use the theory [4-6] based on the multiscale method. The small parameter ε will be defined as

$$\varepsilon^2 = \frac{\mathrm{Ra} - \mathrm{Ra}_0}{\mathrm{Ra}_0}.$$

with $N_{\mathrm{Ra},0}$ denoting the critical value of the Rayleigh number and $\tau = \varepsilon t$. Following the

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procedure in [2], we expand

$$\theta = \theta^{(0)} + \theta^{(1)}E + \theta^{*(1)}E^{-1} + \theta^{(2)}E^2 + \theta^{*(2)}E^{-2} + \ldots, \tag{6}$$

where

$$\theta^{(n)} = \theta^{(n)}(x_3, \tau, \varepsilon);\ \theta^{(1)} = \varepsilon\theta^{(11)} + \varepsilon^2\theta^{(12)} + \varepsilon^3\theta^{(13)} + \ldots;\ \theta^{(0)} = \varepsilon^2\theta^{(02)} +$$
$$+ \varepsilon^3\theta^{(03)} + \ldots;\ \theta^{(2)} = \varepsilon^2\theta^{(22)} + \varepsilon^3\theta^{(23)} + \ldots;\ \theta^{(3)} = \varepsilon^3\theta^{(33)} + \ldots;$$

with the asterisk denoting complex conjugates. The series expansions of u_i and P are analogous.

Inserting expression (6) into Eq. (1)-(2) and equating the terms of like powers in $\varepsilon^n E^m$, we obtain the equations of successive approximations. The εE terms yield the solution according to linear theory. According to [2], we have for $\varepsilon^2 E^0$

$$u_1^{(02)} = u_2^{(02)} = u_3^{(02)} = 0,$$

$$\theta^{(02)} = \frac{\Pr(a_0^2 + \pi^2)|A|^2 \sin 2\pi x_3}{2\pi[\delta_0^2 \Pr + (a_0^2 + \pi^2)^2]},$$

for $\varepsilon^2 E$

$$u_1^{(12)} = u_2^{(12)} = u_3^{(13)} = \theta^{(12)} = 0,$$

and for $\varepsilon^2 E^2$

$$u_1^{(22)} = u_2^{(22)} = u_3^{(22)} = \theta^{(22)} = 0.$$

Let us examine the terms of expansion for $\varepsilon^3 E$

$$S_{ij} = \beta a_{ik} a_{kj} + \gamma(\partial_t a_{ij} + u_j a_{ij,j} + a_{kj} u_{k,j} + a_{ik} u_{k,j}),$$

$$(a_{ik} a_{kj})^{(13)} = (a_{ik}^{(02)} a_{kj}^{(11)} + a_{ik}^{*(11)} a_{kj}^{(22)} + a_{ik}^{(22)} a_{kj}^{*(11)} + a_{ik}^{(11)} a_{kj}^{(02)})\varepsilon^3 E = 0.$$

Analogously, all nonlinear terms in the expression for S_{ij} are zero so that

$$S_{ij}^{(13)} \varepsilon^3 E = \gamma \partial_\tau a_{ij}^{(11)} \varepsilon^3 E.$$

The $\varepsilon^3 E$-order term in the equation of heat conduction is

$$(u_j \theta_{,j})^{(13)} = u_j^{(11)} \theta_{,j}^{(02)}.$$

The $\varepsilon^3 E$ terms in Eqs. (1) and (2) are then

$$\partial_\tau u_i^{(11)} + P_{,i}^{(13)} - \mathrm{Ra}\,\theta^{(13)}\delta_{i3} - u_{i,jj}^{(13)} = \gamma \partial_\tau u_{i,jj}^{(11)},$$
$$\Pr \partial_\tau \theta^{(11)} - \theta_{,jj}^{(13)} - u_3^{(13)} = \Pr u_3^{(11)} \theta_{,j}^{(02)}.$$

The solution is

$$u_3^{(13)} = \mathrm{Ra}_0 a_0^2 (L\partial_\tau A - A + K|A|^2 A) W - Y \cos 3\pi x_3,$$

where

$$L = \frac{\Pr}{i\delta \Pr + a_0^2 + \pi^2} + \frac{(\pi^2 + a_0^2)^2}{\mathrm{Ra}_0 a_0^2} + \gamma \frac{(\pi^2 + a_0^2)^3}{\mathrm{Ra}_0 a_0^2},$$

$$K = \frac{1}{2} \frac{\Pr^2 (a_0^2 + \pi^2)}{\delta^2 \Pr^2 + (a_0^2 + \pi^2)^2}.$$

The condition of orthogonality for $u_3^{(13)}$ to $u_3^{*(11)}$ yields

$$\partial_\tau A = d_1 A + K_1 |A|^2 A, \tag{7}$$

where

$$d_1 = L^{-1};\quad K_1 = -2L^{-1}K.$$

The form of Eq. (7) is the same as that of the corresponding equation in [2]. The coefficients d_1 and K_1 differ from the coefficients in that other equation, because another model of the fluid has been used. We now expand in the vicinity of $i\delta_0$:

$$i\delta = i\delta_0 - iv_1(k - k_0) - iv_2(l - l_0) - a(k - k_0^2) - 2h(l - l_0)(k - k_0) - b(l - l_0) + d_1(\mathrm{Ra} - \mathrm{Ra}_0)\mathrm{Ra}_0^{-1}. \tag{8}$$

This dispersion relation (8) is associated with a wave packet. Here v_1 and v_2 are the group velocities of a wave packet.

We now define new variables

$$\xi = \varepsilon(x_1 - v_1 t), \quad \eta = \varepsilon(x_2 - v_2 t),$$

so that the amplitude equation can be written as

$$\partial_\tau A - a_1 A_{,\xi\xi} - 2hA_{,\xi\eta} - bA_{,\eta\eta} = d_1 A - K_1 |A|^2 A).$$

For the case of steady convection we obtain $v_1 = v_2 = 0$ from the requirement that $i\delta$ be a real quantity, so that

$$d_1 = 3\pi^2 [2(\mathrm{Pr}+1) + 3\pi^2\gamma]^{-1}, \quad K_1 = -\frac{2\mathrm{Pr}^2 d_1}{3\pi^2},$$

$$a_1 = \frac{16}{9\pi^4} d_1 k_0^2, \quad b = \frac{16}{9\pi^4}, \quad h = a_1 b.$$

A few transformations yield finally

$$\partial_\tau A - a_2 \left(A_{,\xi\xi} - \frac{2i}{\pi\sqrt{2}} A_{,\xi\bar\eta\bar\eta} - \frac{1}{2\pi^2} A_{,\bar\eta\bar\eta\bar\eta\bar\eta} \right) = d_1 A + K_1 |A|^2 A, \tag{9}$$

where $\bar\eta = \varepsilon^{1/2} x_2$.

Thus, expression (9) determines the behavior of the perturbation amplitude. The sign of the fast-varying $K_1|A|^2A$ term determines the possible evolution of perturbations. For a Newtonian fluid $K_1 < 0$, i.e., this term decreases fast. With increasing time τ the amplitude approaches its steady-state value

$$A_{ss} = (3\pi^2/2\mathrm{Pr}^2)^{1/2}$$

regardless of whether it was larger or smaller than A_{ss} at instant $\tau = 0$.

For a second-order fluid it is theoretically possible that $d_1 < 0$, and then $K_1 > 0$ at some value of the parameter of the fluid. In this the relation $A(\tau)$ is of a different character than in the case of a Newtonian fluid. It follows from Eq. (9) that a perturbation decays when its amplitude is smaller than A_{ss}. When the amplitude at time $\tau = 0$ is larger than A_{ss}, then the positive cubic term in Eq. (9) makes it increase infinitely "discontinuity-wise." The value of the parameter determining this range is defined by the inequality

$$\gamma < -2(\mathrm{Pr}+1)/3\pi^2.$$

The behavior of the perturbation amplitude is similar and yet different in the case of an Oldroyd fluid within a certain range of parameter values [2], where a "discontinuity-wise" increase of the perturbation amplitude has been found to occur regardless of its initial value.

In real viscoelastic fluids such as polymer solutions it is obviously very difficult to track the behavior of perturbations occurring in a second-order fluid within a definite range of parameter γ. As an example, let us consider an aqueous solution of dextrin with $N_{Pr} \sim 6000$ at $\rho = 1$ g/cm³ [7]. A layer here should be $1.5 \cdot 10^{-3}$ mm thick. This conclusion is analogous to the conclusion about the probability of a vibrational instability, theoretically predicted [1] but hardly possible to confirm experimentally.

NOTATION

τ_{ij}, deviator of the stress tensor; $a_{ik}^{(1)}$, $a_{ij}^{(2)}$, Rivlin–Ericksen tensors; μ, dynamic viscosity; β, γ, material constants of a second-order fluid; ∂_t, operator of differentiation with respect to time; j, a partial derivative with respect to corresponding x_j coordinate; t, time; δ, perturbation decrement; δ_0, critical decrement; θ, temperature perturbation; u_i, velocity; Δ, nabla operator; N_{Pr}, Prandtl number; N_{Ra}, Rayleigh number; $N_{Ra,0}$, critical value of the Rayleigh number; a_0, critical wave number; W, vertical distribution of the perturbation velocity; A, perturbation amplitude; ε, small parameter; *, a complex conjugate; τ, "slow" time; a_{ij}, strain rate tensor; $S_{ij} = \tau_{ij} - a_{ij}$; ∂_τ, operator of differentiation with respect to "slow" time; k_0, l_0, components of the critical wave vector; v_1, v_2, group velocities of a wave packet: a_1, b, h, a_2, d_1, coefficients in the amplitude equation; and A_{ss}, steady-state amplitude.

LITERATURE CITED

1. C. M. Vest and V. S. Arpaci, "Overstability of a viscoelastic fluid layer heated from below," J. Fluid Mech., 36, No. 3, 613-623 (1969).
2. J. A. Eltayeb, "Nonlinear thermal convection in an elasticoviscous layer heated from below," Proc. R. Soc. London, 350, No. 1685, 161-176 (1977).
3. F. A. Garifullin, "Stability of a viscoelastic fluid layer heated from below," Zh. Prikl. Mekh. Tekh. Fiz., No. 6, 55-62 (1974).
4. K. Stewartson and J. T. Stuart, "A nonlinear theory for a wave system in plane Poiseuille flow," J. Fluid Mech., 48, No. 3, 529-545 (1971).
5. L. M. Hocking, S. Stewartson, J. T. Stuart, and S. N. Brown, "A nonlinear instability burst in plane parallel flow," J. Fluid Mech., 51, No. 4, 705-735 (1972).
6. A. Davey, L. M. Hocking, and K. Stewartson, "On the nonlinear evolution of three-dimensional disturbances in plane Poiseuille flow," J. Fluid Mech., 63, No. 3, 529-536 (1974).
7. C. Bruce and W. H. Schwarz, "An experimental study of second- and third-order fluids," Am. Chem. Soc., Polym. Prepr., 10, 152-159 (1969).

STABILITY OF THE FLOW OF A ROTATING LIQUID FILM ALONG THE INSIDE SURFACE OF A CYLINDRICAL TUBE

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Under consideration is the problem of formation of Taylor vortices in a rotating film of a viscous incompressible fluid.

Recently, film flow of a rotating liquid has become used on a wide scale in diverse equipment of the chemical industry (evaporators, heat exchangers, chemical reactors). Superposition of rotation on gravity flow of a film facilitates uniform spraying of the equipment surface, especially at low rates of liquid consumption, and appreciably intensifies heat and mass transfer processes [1-3]. The principal parameter determining this intensification is the stream whirl factor numerically equal to the tangent of the angle between the line of flow and the generatrix of the tube. It has been established [2] that with $\tan\beta \sim 1$ a heat transfer coefficient more than twice as high as during gravity flow of the film is attainable. This intensification effect weakens as $\tan\beta$ decreases and, when $\tan\beta < 0.1$, it becomes negligible. Therefore, selection of the optimum tube height for imparting rotation to a film is one of the more important problems in rational equipment design. As is well known, under such conditions a film unwhirls along the height, because of friction at a solid surface, and $\tan\beta$ decreases correspondingly. This decrease along the tube height is not monotonic, however, and experiments have revealed [3] that at a certain spray density the $\tan\beta$ curve begins to break at some point, with the rate of change of $\tan\beta$ much higher along the initial segment than beyond this break point. This trend is illustrated graphically in Fig. 1: Experimental data are shown here obtained in another study [4] with flow of a water film along the surface of a tube $3\cdot10^{-2}$ m in diameter at a temperature of 19°C.

Exponential decreasing of $\tan\beta$ along the tube height has been established theoretically [3] and confirmed experimentally, as shown in Fig. 1. As to the break point and the corresponding change of the attenuation rate (at $\beta > \beta_{cr}$ the whirl factor decreases approximately 5.7 times faster), no satisfactory explanation of this phenomenon has yet been found. Meanwhile, determination of the critical $\tan\beta$ corresponding to the break point on a $\tan\beta = g(z/R)$ curve is of great practical importance, because maximum intensification of the transfer processes can be expected to occur within the initial range.

A break point on the $\tan\beta$ curve can be regarded as a consequence of a substantial change in the conditions of flow and, particularly, loss of stability so that a solution to the equations of laminar flow would not reveal it. Such a phenomenon is generally characteristic of flow of a liquid in the field of centrifugal forces. In the case of flow of a liquid between

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